10.A **nondimensional natural coordinates** in the one-dimensional elements

특성좌표계 : 무차원화된 좌표계 (표준좌표계)

뒤틀린 요소의 를 용이하게 표현

개념을 이해하기 쉽고, 수치적분을 사용하면 프로그램 구성이 용이.

(10.7) $U(x) = \{\phi(x)\}^T\{a\}$ $\Rightarrow$ $U(\xi) = \{L(\xi)\}^T\{a\}$

$(j-1)h < x < jh$ $\qquad$ $-1 < \xi < 1$

<u>shape functions</u>

linear element (one-degree element)

h: x_1 —— x_2 $\Rightarrow$ 1 —— 2 ; -1 $\mapsto\xi$ 1

$L_1 = \frac{1}{2}(1-\xi)$ $\qquad$ $L_2 = \frac{1}{2}(1+\xi)$

quadratic element (two-degree element)

x_1 —— x_2 —— x_3 $\Rightarrow$ 1 —— 2 —— 3 ; -1 $\quad 0$ $\quad 1$

$L_1 = -\frac{1}{2}\xi(1-\xi)$ $\qquad$ $L_2 = (1+\xi)(1-\xi)$ $\qquad$ $L_3 = \frac{1}{2}\xi(1+\xi)$

cubic element (three-degree element)

x_1 —— x_2 —— x_3 —— x_4 $\Rightarrow$ 1 —— 2 —— 3 —— 4 ; -1 $\quad -⅓$ $\quad ⅓$ $\quad 1$

$L_1 = -\frac{9}{16}(\xi-1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})$ $\qquad$ $L_2 = \frac{27}{16}(\xi+1)(\xi-1)(\xi-\frac{1}{3})$

$L_3 = -\frac{27}{16}(\xi+1)(\xi-1)(\xi+\frac{1}{3})$ $\qquad$ $L_4 = \frac{9}{16}(\xi+1)(\xi+\frac{1}{3})(\xi-\frac{1}{3})$

(10.15) $\rightarrow$ $[k] = \int_{x_1}^{x_2} EA(x)\{L'(x)\}\{L'(x)\}^T dx$

(10.18) $\rightarrow$ $[m] = \int_{x_1}^{x_2} m(x)\{L(x)\}\{L(x)\}^T dx$

<u>derivative of shape functions</u> in terms of the natural coordinates

$$\frac{dL_i}{dx} = \frac{dL_i}{d\xi}\frac{d\xi}{dx} \qquad (i = 1,\ 2,\ \cdots,\ r) \qquad \frac{d\xi}{dx} = ?$$

r : number of nodes = in a linear element

in a quadratic element

in a cubic element

$$L_i = L_i(\xi) \quad \Rightarrow \quad \frac{dL_i}{d\xi}$$

$$x = \sum_{i=1}^{r} L_i(\xi)\ x_i = x(\xi) \quad \Rightarrow \quad \frac{dx}{d\xi}$$

$$\frac{dL_i}{d\xi} = \frac{dL_i}{dx}\frac{dx}{d\xi} \qquad \frac{dL_i}{dx} = ?$$

in the matrix form $\left\{\frac{dL}{d\xi}\right\} = \left\{\frac{dL}{dx}\right\}\left[\frac{dx}{d\xi}\right]$, $\left[\frac{dx}{d\xi}\right] = [J]$:

$r\times1 \quad r\times1 \quad 1\times1$ in one-dimension

$$\Rightarrow \left\{\frac{dL}{dx}\right\} = \left\{\frac{dL}{d\xi}\right\}\left[\frac{dx}{d\xi}\right]^{-1} =$$

$$dx = \left|\frac{dx}{d\xi}\right| d\xi = \det[J]\ d\xi$$

$$[k] = EA\int_{x_1}^{x_2}\left\{\frac{dL}{dx}\right\}\left\{\frac{dL}{dx}\right\}^T dx$$

$$= EA\int_{-1}^{1}\left\{\frac{dL}{d\xi}\right\}\left[\frac{dx}{d\xi}\right]^{-1}\left(\left[\frac{dx}{d\xi}\right]^{-1}\right)^T\left\{\frac{dL}{d\xi}\right\}^T\left|\frac{dx}{d\xi}\right| d\xi$$

ex. linear element

$$L_1 = \frac{1}{2}(1-\xi) \quad \Rightarrow \quad \frac{dL_1}{d\xi} =$$

$$L_2 = \frac{1}{2}(1+\xi) \quad \Rightarrow \quad \frac{dL_2}{d\xi} =$$

$$x = L_1\,x_1 + L_2\,x_2 \quad \Rightarrow \quad \frac{dx}{d\xi} = \frac{dL_1}{d\xi}x_1 + \frac{dL_2}{d\xi}x_2 = -\frac{1}{2}x_1 + \frac{1}{2}x_2$$

$$= \frac{x_2 - x_1}{2} =$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = EA\int_{-1}^{1}\left\{\frac{dL}{d\xi}\right\}\left[\frac{dx}{d\xi}\right]^{-1}\left(\left[\frac{dx}{d\xi}\right]^{-1}\right)^T\left\{\frac{dL}{d\xi}\right\}^T\left|\frac{dx}{d\xi}\right| d\xi$$

$$=$$

10.B Two-Dimensional Finite Elements

(reference: 임상전 등 공저, 유한요소법입문, 동명사, 1993.)

interpolation polynomials (interpolation functions, shape functions)

연속함수를 유한개의 요소에서의 부분연속함수의 집합으로서 근사적으로 표현하는 가장 일반적인 형태

형상함수를 다항식의 형태로 나타낼 때, 상수항을 포함한 항의 수는
요소의 전체 자유도와 같아야 한다.

= 절점 수 × 절점당 자유도 (DOF)

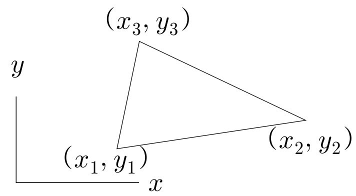
스칼라 양 : 자유도 (ex.
2차원 변위 : 자유도 (
3차원 변위 : 자유도 (
3차원 일반운동 : 자유도 (

linear elements

선형요소 = 1차요소 (one-degree element)

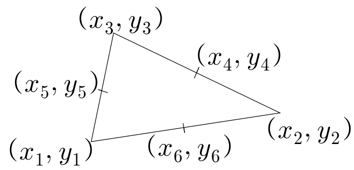
: 단순(simple)요소

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y$$



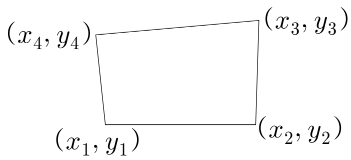
cf. 복잡(complex)요소

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$



복합(multiplex)요소 (선형요소로 간주)

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$



one-DOF linear elements

1자유도 영역변수 ϕ

(i) one-dimensional element

양단에서 절점(1,2)를 갖는 2절점 선 요소 (길이 h)

절점 좌표 $0,\ h$

절점에서 영역변수 값 $\phi_1,\ \phi_2$

$$\phi(x) = \alpha_1 + \alpha_2 x$$

$$\phi(0) = \alpha_1 = \phi_1 \qquad \phi(h) = \alpha_1 + \alpha_2 h = \phi_2 \quad \Rightarrow \quad \alpha_2 = \frac{\phi_2 - \phi_1}{h}$$

$$\phi(x) = \phi_1 + \frac{\phi_2 - \phi_1}{h} x = (1-\frac{x}{h})\ \phi_2 + \frac{x}{h}\ \phi_1$$

$$\phi = L_1 \phi_1 + L_2 \phi_2 = [L]\ \{\phi\}_e$$

$$\{\phi\}_e = \{\phi_1 \quad \phi_2\}^T$$

$$[L] = [L_1 \quad L_2] = [(1-\frac{x}{h}) \quad \frac{x}{h}]$$: shape functions

(ii) two-dimensional element (triangular element)

꼭지점에서 절점(1, 2, 3)을 갖는 3절점 삼각형 요소

절점 좌표 $(x_1,\ y_1),\ (x_2,\ y_2),\ (x_3,\ y_3)$

절점에서 영역변수 값 $\phi_1,\ \phi_2,\ \phi_3$

$$\phi(x,y) = \alpha_1 + \alpha_2\, x + \alpha_3\, y$$

$$\phi(x_1,y_1) = \alpha_1 + \alpha_2\, x_1 + \alpha_3\, y_1 = \phi_1$$
$$\phi(x_2,y_2) = \alpha_1 + \alpha_2\, x_2 + \alpha_3\, y_2 = \phi_2 \qquad D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$
$$\phi(x_3,y_3) = \alpha_1 + \alpha_2\, x_3 + \alpha_3\, y_3 = \phi_3$$

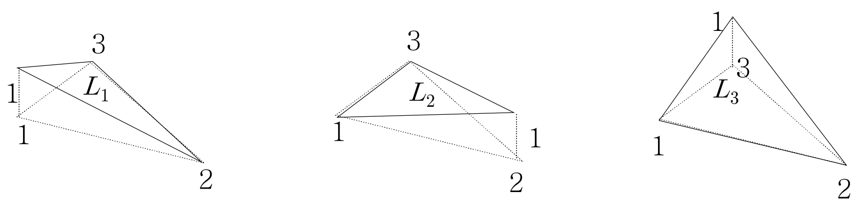
$$\alpha_1 = \frac{1}{D}\begin{vmatrix} \phi_1 & x_1 & y_1 \\ \phi_2 & x_2 & y_2 \\ \phi_3 & x_3 & y_3 \end{vmatrix} = \frac{1}{D}[(x_2 y_3 - x_3 y_2)\phi_1 + (x_3 y_1 - x_1 y_3)\phi_2 + (x_1 y_2 - x_2 y_1)\phi_3] = \frac{1}{D}(a_1\,\phi_1 + a_2\,\phi_2 + a_3\,\phi_3)$$

$$\alpha_2 = \frac{1}{D}\begin{vmatrix} 1 & \phi_1 & y_1 \\ 1 & \phi_2 & y_2 \\ 1 & \phi_3 & y_3 \end{vmatrix} = \frac{1}{D}[(y_2 - y_3)\phi_1 + (y_3 - y_1)\phi_2 + (y_1 - y_2)\phi_3] = \frac{1}{D}(b_1\,\phi_1 + b_2\,\phi_2 + b_3\,\phi_3)$$

$$\alpha_3 = \frac{1}{D}\begin{vmatrix} 1 & x_1 & \phi_1 \\ 1 & x_2 & \phi_2 \\ 1 & x_3 & \phi_3 \end{vmatrix} = \frac{1}{D}[(x_3 - x_2)\phi_1 + (x_1 - x_3)\phi_2 + (x_2 - x_1)\phi_3] = \frac{1}{D}(c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3)$$

$$\phi = \frac{1}{D}(a_1\,\phi_1 + a_2\,\phi_2 + a_3\,\phi_3) + \frac{1}{D}(b_1\,\phi_1 + b_2\,\phi_2 + b_3\,\phi_3)\ x + \frac{1}{D}(c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3)\ y$$
$$= \frac{1}{D}(a_1 + b_1\,x + c_1\,y)\ \phi_1 + \frac{1}{D}(a_2 + b_2\,x + c_2\,y)\ \phi_2 + \frac{1}{D}(a_3 + b_3\,x + c_3\,y)\ \phi_3$$
$$= L_1\ \phi_1 + L_2\ \phi_2 + L_3\ \phi_3$$

$L_1(x_1,y_1) =$	$L_2(x_1,y_1) =$	$L_3(x_1,y_1) =$
$L_1(x_2,y_2) =$	$L_2(x_2,y_2) =$	$L_3(x_2,y_2) =$
$L_1(x_3,y_3) =$	$L_2(x_3,y_3) =$	$L_3(x_3,y_3) =$



영역변수 $\phi(x,y)$의 도함수

$$\phi(x,y) = L_1(x,y)\ \phi_1 + L_2(x,y)\ \phi_2 + L_3(x,y)\ \phi_3$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial L_1}{\partial x}\phi_1 + \frac{\partial L_2}{\partial x}\phi_2 + \frac{\partial L_3}{\partial x}\phi_3 = \frac{1}{D}(b_1\,\phi_1 + b_2\,\phi_2 + b_3\,\phi_3)$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial L_1}{\partial y}\phi_1 + \frac{\partial L_2}{\partial y}\phi_2 + \frac{\partial L_3}{\partial y}\phi_3 = \frac{1}{D}(c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3)$$

(iii) three-dimensional element (skip)

<u>multi-DOF two-dimensional linear elements</u>

(i) two-DOF element

2자유도 영역변수 u, v

$$u = L_1 u_1 + L_2 u_2 + L_3 u_3 = L_1 u_1 + 0\ v_1 + L_2 u_2 + 0\ v_2 + L_3 u_3 + 0\ v_3$$

$$v = L_1 v_1 + L_2 v_2 + L_3 v_3 = 0\ u_1 + L_1 v_1 + 0\ u_2 + L_2 v_2 + 0\ u_3 + L_3 v_3$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} L_1 & 0 & L_2 & 0 & L_3 & 0 \\ 0 & L_1 & 0 & L_2 & 0 & L_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(ii) three-DOF element

3자유도 영역변수 u, v, w

$$u = L_1 u_1 + L_2 u_2 + L_3 u_3$$

$$v = L_1 v_1 + L_2 v_2 + L_3 v_3$$

$$v = L_1 w_1 + L_2 w_2 + L_3 w_3$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 & 0 \\ 0 & L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 \\ 0 & 0 & L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \end{Bmatrix}$$

※ 수렴 조건

유한요소법에 의한 근사해가 정해에 수렴하기 위해서는
근사함수가 완전성, 적합성, 기하학적 등방성을 만족해야 한다.

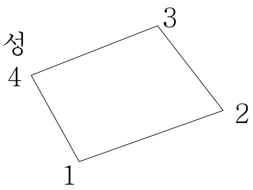
완전성 (완전조건)

모든 형상함수의 합이 1 $\sum_{i=1}^{N} L_i = 1$ ex. $L_1 = 1 - \frac{x}{h}$, $L_2 = \frac{x}{h}$

적합성 (적합조건)

모델의 연속 - 요소 경계에서 절점 일치
영역변수의 연속 - 미분방정식이 n계이면 $(n-1)$계 도함수가 유한, 연속이어야 함

기하학적 등방성



1-2 가 선형 형상함수이면 2-3 도 선형 형상함수

higher-degree elements

형상함수의 차수가 높아지면

같은 정확도의 근사해를 구하는 데에 요구되는 요소의 수

유한요소 방정식을 풀어서 해를 구하는 계산시간

요소의 계수 행렬과 외력벡터 수치를 계산하는 시간은

따라서 문제의 성격과 목적에 따라 적절하게 선택해야 하므로,

경험을 통한 공학적 판단이 요구됨.

tetragonal element (two-dimensional element)

(i) linear element (one-degree element)

$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x y$

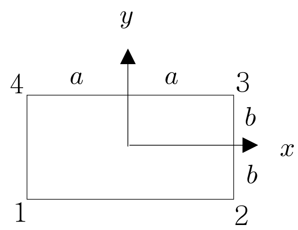
$\phi(-a, -b) = \alpha_1 - a\,\alpha_2 - b\,\alpha_3 + a\,b\,\alpha_4 = \phi_1$

$\phi(a, -b) = \alpha_1 + a\,\alpha_2 - b\,\alpha_3 - a\,b\,\alpha_4 = \phi_2$

$\phi(a, b) = \alpha_1 + a\,\alpha_2 + b\,\alpha_3 + a\,b\,\alpha_4 = \phi_3$

$\phi(-a, b) = \alpha_1 - a\,\alpha_2 + b\,\alpha_3 - a\,b\,\alpha_4 = \phi_4$

$\Rightarrow$



$\phi(x, y) = \alpha_1(\phi_1,\phi_2,\phi_3,\phi_4) + \alpha_2(\phi_1,\phi_2,\phi_3,\phi_4)\ x + \alpha_3(\phi_1,\phi_2,\phi_3,\phi_4)\ y + \alpha_4(\phi_1,\phi_2,\phi_3,\phi_4)\ xy$

$= L_1(x, y)\ \phi_1 + L_2(x, y)\ \phi_2 + L_3(x, y)\ \phi_3 + L_4(x, y)\ \phi_4$

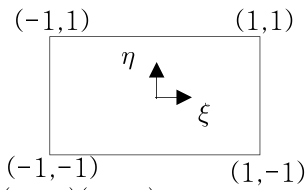
$L_1(x, y) = \frac{1}{4ab}(a-x)(b-y)$ $\qquad L_2(x, y) = \frac{1}{4ab}(a+x)(b-y)$

$L_3(x, y) = \frac{1}{4ab}(a+x)(b+y)$ $\qquad L_4(x, y) = \frac{1}{4ab}(a-x)(b+y)$

$L_1 + L_2 + L_3 + L_4 =$

in terms of the natural coordinates

$\xi = \frac{x}{a},\ \eta = \frac{y}{b}$

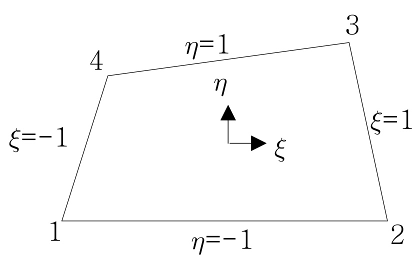


$L_1(\xi,\ \eta) = \frac{1}{4ab}(a - a\xi)(b - b\eta)$

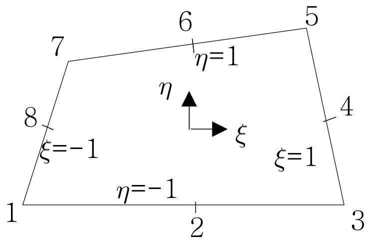
$= \frac{1}{4}(1-\xi)(1-\eta),$ $\qquad L_2(\xi,\ \eta) = \frac{1}{4}(1+\xi)(1-\eta),$

$L_3(\xi,\ \eta) = \frac{1}{4}(1+\xi)(1+\eta),$ $\qquad L_4(\xi,\ \eta) = \frac{1}{4}(1-\xi)(1+\eta)$

arbitrary tetragonal element



(ii) quadratic element (two-degree element)



$$\phi(x,y) = \alpha_1 + \alpha_2\,\xi + \alpha_3\,\eta + \alpha_4\,\xi\eta + \alpha_5\,\xi^2 + \alpha_6\,\eta^2 + \alpha_7\,\xi^2\eta + \alpha_8\,\xi\eta^2$$

$$= L_1(\xi,\eta)\,\phi_1 + L_2(\xi,\eta)\,\phi_2 + L_3(\xi,\eta)\,\phi_3 + L_4(\xi,\eta)\,\phi_4 + L_5(\xi,\eta)\,\phi_5 + L_6(\xi,\eta)\,\phi_6 + L_7(\xi,\eta)\,\phi_7 + L_8(\xi,\eta)\,\phi_8$$

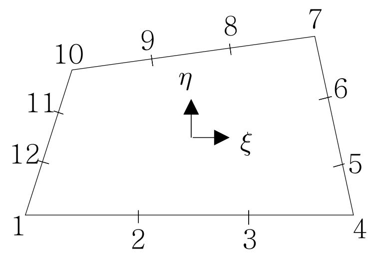
$L_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1)$, $L_2(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$

$L_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$, $L_4(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$

$L_5(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$, $L_6(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$

$L_7(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(\xi-\eta+1)$, $L_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$

(iii) cubic element (three-degree element)



$$\phi(x,y) = \alpha_1 + \alpha_2\,\xi + \alpha_3\,\eta + \alpha_4\,\xi\eta + \alpha_5\,\xi^2 + \alpha_6\,\eta^2 + \alpha_7\,\xi^3 + \alpha_8\,\xi^2\eta + \alpha_9\,\xi\eta^2 + \alpha_{10}\,\eta^3 + \alpha_{11}\,\xi^3\eta + \alpha_{12}\,\xi\eta^3$$

$$= L_1(\xi,\eta)\,\phi_1 + L_2(\xi,\eta)\,\phi_2 + L_3(\xi,\eta)\,\phi_3 + L_4(\xi,\eta)\,\phi_4 + L_5(\xi,\eta)\,\phi_5 + L_6(\xi,\eta)\,\phi_6 + L_7(\xi,\eta)\,\phi_7 + L_8(\xi,\eta)\,\phi_8 + L_9(\xi,\eta)\,\phi_9 + L_{10}(\xi,\eta)\,\phi_{10} + L_{11}(\xi,\eta)\,\phi_{11} + L_{12}(\xi,\eta)\,\phi_{12}$$

$L_1(\xi,\eta) = \frac{1}{32}(1-\xi)(1-\eta)[-10+9(\xi^2+\eta^2)]$, $L_2(\xi,\eta) = \frac{9}{32}(1-\xi^2)(1-\eta)(1-3\xi)$

$L_3(\xi,\eta) = \frac{9}{32}(1-\xi^2)(1-\eta)(1+3\xi)$, $L_4(\xi,\eta) = \frac{1}{32}(1+\xi)(1-\eta)[-10+9(\xi^2+\eta^2)]$

$L_5(\xi,\eta) = \frac{9}{32}(1+\xi)(1-\eta^2)(1-3\eta)$, $L_6(\xi,\eta) = \frac{9}{32}(1+\xi)(1-\eta^2)(1+3\eta)$

$L_7(\xi,\eta) = \frac{1}{32}(1+\xi)(1+\eta)[-10+9(\xi^2+\eta^2)]$, $L_8(\xi,\eta) = \frac{9}{32}(1-\xi^2)(1+\eta)(1+3\xi)$

$L_9(\xi,\eta) = \frac{9}{32}(1-\xi^2)(1+\eta)(1-3\xi)$, $L_{10}(\xi,\eta) = \frac{1}{32}(1-\xi)(1+\eta)[-10+9(\xi^2+\eta^2)]$

$L_{11}(\xi,\eta) = \frac{9}{32}(1-\xi)(1-\eta^2)(1+3\eta)$, $L_{12}(\xi,\eta) = \frac{9}{32}(1-\xi)(1-\eta^2)(1-3\eta)$

$$[\,m\,] = \int\int_A m(x,y)\,\{L(x,y)\}\,\{L(x,y)\}^T\,dx\,dy$$

$$[\,k\,] = \int\int_A EA(x,y)\,\{L'(x,y)\}\,\{L'(x,y)\}^T\,dx\,dy$$

<u>derivatives of shape functions</u> in two-dimensional elements

$$\frac{dL_i}{dx} = \frac{\partial L_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial L_i}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \frac{dL_i}{dy} = \frac{\partial L_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial L_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{dL_i}{d\xi} = \frac{\partial L_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial L_i}{\partial y}\frac{\partial y}{\partial \xi} \qquad \frac{dL_i}{d\eta} = \frac{\partial L_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial L_i}{\partial y}\frac{\partial y}{\partial \eta}$$

in the matrix form

$$\left[\left\{\frac{\partial L}{\partial \xi}\right\}\left\{\frac{\partial L}{\partial \eta}\right\}\right] = \left[\left\{\frac{\partial L}{\partial x}\right\}\left\{\frac{\partial L}{\partial y}\right\}\right]\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{bmatrix}, \quad \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{bmatrix} =$$

$r \times 2$ $\qquad$ $r \times 2$ $\qquad$ 2×2 $\qquad$ in two-dimension

$$\Rightarrow \left[\left\{\frac{\partial L}{\partial x}\right\}\left\{\frac{\partial L}{\partial y}\right\}\right] = \left[\left\{\frac{\partial L}{\partial \xi}\right\}\left\{\frac{\partial L}{\partial \eta}\right\}\right]\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{bmatrix}^{-1} = \left[\left\{\frac{\partial L}{\partial \xi}\right\}\left\{\frac{\partial L}{\partial \eta}\right\}\right][\,J\,]^{-1}$$

$$dx\,dy = \det\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{vmatrix}\,d\xi\,d\eta =$$

$$x(\xi,\eta) = \sum_{i=1}^{r} L_i(\xi,\eta)\;x_i \qquad y(\xi,\eta) = \sum_{i=1}^{r} L_i(\xi,\eta)\;y_i$$

r : number of nodes = in a linear element
in a quadratic element
in a cubic element

$$J_{11} = \frac{dx}{d\xi} = \sum_{i=1}^{r}\frac{\partial L_i}{\partial \xi}x_i \qquad J_{12} = \frac{dx}{d\eta} = \sum_{i=1}^{r}\frac{\partial L_i}{\partial \eta}x_i$$

$$J_{21} = \frac{dy}{d\xi} = \sum_{i=1}^{r}\frac{\partial L_i}{\partial \xi}y_i \qquad J_{22} = \frac{dy}{d\eta} = \sum_{i=1}^{r}\frac{\partial L_i}{\partial \eta}y_i$$

$$[\,k\,] = EA\int\int_A \left[\left\{\frac{\partial L}{\partial \xi}\right\}\left\{\frac{\partial L}{\partial \eta}\right\}\right][J]^{-1}\left([J]^{-1}\right)^T\left[\left\{\frac{\partial L}{\partial \xi}\right\}\left\{\frac{\partial L}{\partial \eta}\right\}\right]^T \det[J]\;d\xi\,d\eta$$

FEM (finite-element method)

approximate method to solve the differential equations in boundary-value problems or initial-value problems.

ex.

CAE (Computer-Aided Engineering)