6.9 Waves in a rod of circular cross-section

cylindrical coordinates :

displacements :

equations of motion

$$\nabla^2 u - \cdots = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2} \tag{6.92}$$

$$\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{1-2\nu}\frac{1}{r}\frac{\partial \Delta}{\partial \theta} = \frac{1}{c_T^2}\frac{\partial^2 v}{\partial t^2} \tag{6.93}$$

$$\nabla^2 w + \cdots = \frac{1}{c_T^2}\frac{\partial^2 w}{\partial t^2} \tag{6.94}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad : \tag{6.95}$$

$$\triangle = \nabla\cdot(u\ \mathrm{i}_r + v\ \mathrm{i}_\Theta + w\ \mathrm{i}_z) = \frac{\partial u}{\partial r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right) + \frac{\partial w}{\partial z} \quad : \tag{6.96}$$

axisymmetric (axially symmetric) → §6.10

§2.13 summary of equations in cylindrical coordinates

displacement potentials

(2.132) $\quad u = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z}$

(2.133) $\quad v = \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}$

(2.134) $\quad w = \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial(\psi_\theta r)}{\partial r} - \frac{1}{r}\frac{\partial \psi_r}{\partial \theta}$

equations of motion

(2.135) $\quad \nabla^2 \phi = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}$ (6.100)

(2.139) $\quad \nabla^2 \psi_z = \frac{1}{c_T^2}\frac{\partial^2 \psi_z}{\partial t^2}$ (6.101)

(2.137) $\quad \nabla^2 \psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial \psi_\theta}{\partial \theta} = \frac{1}{c_T^2}\frac{\partial^2 \psi_r}{\partial t^2}$ (6.102)

(2.138) $\quad \nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \psi_r}{\partial \theta} = \frac{1}{c_T^2}\frac{\partial^2 \psi_\theta}{\partial t^2}$ (6.103)

boundary conditions

(2.144) $\quad \sigma_r = \lambda \triangle + 2G\frac{\partial u}{\partial r} =$ at $r = a$ (6.97) (6.104a)

(2.147) $\quad \tau_{r\theta} = G\left[\frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \frac{\partial v}{\partial r}\right] =$ (6.98) (6.104b)

(2.149) $\quad \tau_{rz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) =$ (6.99) (6.104c) at $r = 0$, (solid cross-section)

at $r = r_i$, (hollow cross-section)

solutions of the form (

$\phi(r, \theta, z, t) = \Phi(r)\ \Theta(\theta)\ \exp[i(kz - \omega t)]$ ⇒ (6.105)~(6.107) (6.116)

$\psi_z(r, \theta, z, t) = \Psi_z(r)\ \Theta(\theta)\ \exp[i(kz - \omega t)]$ ⇒ (6.108) (6.117)

$\psi_r(r, \theta, z, t) = \Psi_r(r)\ \Theta(\theta)\ \exp[i(kz - \omega t)]$ ⎤ (6.109) (6.113) (6.118)

⇒ (6.111-112) (6.115)

$\psi_\theta(r, \theta, z, t) = \Psi_\theta(r)\ \Theta(\theta)\ \exp[i(kz - \omega t)]$ ⎦ (6.110) (6.114) (6.119)

6.10 Waves in a circular rod of solid cross-section

motions which are independent of θ but do depend on z (

- torsional waves
- longitudinal waves

motions which depend on both z and

- flexural waves

6.10.1 torsional waves

circumferential displacement independent of θ $\Rightarrow$ $v(r, z, t)$

equation of motion

(6.93)→ $\nabla^2 v - \frac{v}{r^2} = \frac{1}{c_T^2}\frac{\partial^2 v}{\partial t^2}$, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{c_T^2}\frac{\partial^2 v}{\partial t^2} \tag{6.124}$$

if not circular →

separation of variables

$v(r, z, t) =$

$R'' + \frac{1}{r}R' - \frac{1}{r^2}R - k^2 R = -\frac{\omega^2}{c_T^2}R$

(i) if $k = \omega / c_T$

$R'' + \frac{1}{r}R' - \frac{1}{r^2}R = 0$

$R(r) =$ R is at $r = 0$ →

$$v(r, z, t) = A\, r\, \exp[i(kz - \omega t)] \tag{6.127}$$

satisfies $\tau_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) =$ at $r = a$ automatically

lowest torsional mode $c = \omega / k =$

(ii) if $k \neq \omega / c_T$

$$r^2 R'' + r R' + (q^2 r^2 - 1)R = 0 \qquad q^2 = \frac{\omega^2}{c_T^2} - k^2 \tag{6.128}$$

$R(r) = A\, J_1(qr) + B\, Y_1(qr)$ R is finite (or 0) at $r = 0$ →

$$v(r, z, t) = A\, J_1(qr)\, \exp[i(kz - \omega t)] \tag{6.125}$$

$\tau_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = 0$ at $r = a$

$$\frac{d}{dr}[J_1(qr)] = q\, J_0(qr) - \frac{1}{r}J_1(qr) = -q\, J_2(qr) + \frac{1}{r}J_1(qr)$$

$$J_2(qa) = 0 \qquad qa = \tag{6.126}$$

higher torsional modes (

6.10.2 longitudinal waves

displacement components in the radial and axial directions

independent of θ $\Rightarrow$

$$u = \frac{\partial \phi}{\partial r} - \frac{\partial \psi_\theta}{\partial z} \ , \qquad w = \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial(\psi_\theta r)}{\partial r} \qquad \rightarrow$$

equations of motion

$$(6.102) \ \rightarrow \ \nabla^2 \phi = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2} \qquad \Rightarrow \qquad \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}$$

$$(6.103) \ \rightarrow \ \nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} = \frac{1}{c_T^2}\frac{\partial^2 \psi_\theta}{\partial t^2} \qquad \Rightarrow \qquad \frac{\partial^2 \psi_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r^2} + \frac{\partial^2 \psi_\theta}{\partial z^2} = \frac{1}{c_T^2}\frac{\partial^2 \psi_\theta}{\partial t^2}$$

separation of variables and solutions

$$\phi(r, z, t) = \Phi(r) \ \exp[i\,(k z - \omega t)]$$

$$= A \ J_0(pr) \ \exp[i\,(k z - \omega t)] \qquad p^2 = \frac{\omega^2}{c_L^2} - k^2$$

$$\psi_\theta(r, z, t) = \Psi(r) \ \exp[i\,(k z - \omega t)]$$

$$= C\, J_1(qr) \ \exp[i\,(k z - \omega t)] \qquad q^2 = \frac{\omega^2}{c_T^2} - k^2$$

$$u(r, z, t) = \qquad (6.129)$$

$$w(r, z, t) = \qquad (6.130)$$

boundary conditions

$$\sigma_r = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) + 2\,G\frac{\partial u}{\partial r} = \qquad \text{at} \quad r = a$$

$$\Rightarrow \ \left[-\frac{1}{2}(q^2 - k^2)\,J_0(pa) + \frac{p}{a}J_1(pa)\right]A + \left[-i\,k\,q\,J_0(qa) + \frac{ik}{a}J_1(qa)\right]C = \qquad \cdots①$$

$$\tau_{rz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = \qquad \text{at} \quad r = a$$

$$\Rightarrow \ [-2\,i\,k\,p\,J_1(pa)]\,A - (q^2 - k^2)\,J_1(qa)\,C = \qquad \cdots②$$

①, ② $\Rightarrow$ nontrivial condition for A, C

$$\frac{2p}{a}(q^2 + k^2)\,J_1(pa)\,J_1(qa) - (q^2 - k^2)^2\,J_0(pa)\,J_1(qa) - 4\,k^2\,p\,q\,J_1(pa)\,J_0(qa) = \qquad (6.131)$$

Pochhammer frequency equation

Fig. 6.13 dimensionless frequencies $(\omega a/\pi c_T)$ vs. dimensionless wavenumber (ka/π)
Fig. 6.14 dimensionless phase velocity (c/c_b) vs. " " $(ka/2\pi)$
Fig. 6.15 dimensionless group velocity (c_g/c_b) vs. " " $(ka/2\pi)$

ex.

6.10.3 flexural waves

non-axisymmetric motion (skip)

6.11 Approximate theories for rods

wave motions in rods of arbitrary cross-sections can be described by one-dimensional approximate theories.

ex.

6.11.1 longitudinal motions

assumption : Cross-sectional area of the rod remains

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c_b^2}\frac{\partial^2 u}{\partial t^2} \qquad c_b^2 =$$

Axial shear modes and radial modes are

6.11.2 torsional motions

assumption : Transverse sections remain

The motion consists of a rotation of the sections about the axis.

(Both the warping and the in-plane motions are neglected)

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \qquad c^2 =$$

J : torsional constant (GJ :

I : moment of inertia ($J = I$ for a circular cross-section)

radius of gyration k ⇒ $I =$

$$c^2 = \frac{D}{\rho A k^2} \quad \Rightarrow \quad c = \sqrt{\frac{D}{\rho A}}\,\frac{1}{k} \tag{6.142}$$

6.11.3 flexural motions

assumption :

1. Dominant displacement component is parallel to the plane of symmetry.
2. Deflections are small and cross-sectional areas remain plane and normal to the neutral axis.

$$\frac{\partial^2 w}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} = 0 \tag{6.143}$$

A : cross-sectional area

I : second moment of the cross-sectional area about the neutral axis

$$w(x,t) = B\exp[ik(x-ct)] \quad \Rightarrow \quad -k^2c^2 + k^4\frac{EI}{\rho A} = 0 \quad \Rightarrow \quad c^2 =$$

$$\Rightarrow \quad c = \sqrt{\frac{E}{\rho}}\sqrt{\frac{I}{A}}\,k \tag{6.144}$$

$c \propto k$ (phase velocity is proportional to the wavenumber)

correct for small wavenumbers (long waves)

for a circular cylindrical rod ($I =$ $\Rightarrow \frac{I}{A} = \frac{1}{4}$)

$$c = \frac{1}{2}\sqrt{\frac{E}{\rho}}\,k \tag{6.145}$$

6.12 Approximate theories for plates (skip)

6.12.1 flexural motions - classical theory

6.12.2 effects of transverse shear and rotary inertia

6.12.3 extensional motions