연 속 계 진 동 해 석 중 간 시 험

[30 점]

대학원 기계공학과 2013. 10. 19.

1.[8점] Consider the torsional vibration of a uniform circular shaft fixed at one end x=0 and free at the other end x=L.

(a) Derive the expressions for the natural modes and frequencies.

(b) Verify that the eigenfunctions obtained in (a) are orthogonal with respect to the mass density.

2.[8점] 균일한 보(beam)의 굽힘 진동(bending vibration)을 횡방향 변위(transverse displacement) $y(x,t)$로 표현한다. 이 막대는 길이가 L이고 단위길이당 질량이 m이며, 굽힘 강성(flexural rigidity)이 EI이고, 외력은 작용하지 않는다. x=0인 지점에서 단순지지(pinned) 되어 있고 x=L인 지점에서 자유롭다.

(a) Newton의 운동 제2법칙에 근거하여 differential equation of motion을 유도하고, 경계조건을 제시하시오.

(b) 변수분리(separation of variation)법을 적용하여 특성방정식(characteristic equation)을 구하시오.

(c) 최저차 모드 3개의 고유진동수(natural frequency)와 모드형상(mode shape)식을 구하시오.

3.[8점] 균일한 막대(rod)의 종(axial)방향 자유진동을 고려한다. 이 막대는 길이가 L이고, 단위길이당 질량이 m이며, 탄성계수가 E이며, 단면적이 A이다. 한쪽 끝(x=0)에서 고정되어 있고, 다른 쪽 끝(x=L)에서 자유롭다. 종방향 변위 $u(x,t)$로 표현된 운동방정식이 다음과 같다.

$$EA\,\frac{\partial^2 u(x,t)}{\partial x^2} = m\,\frac{\partial^2 u(x,t)}{\partial t^2}$$

eigenvalue problem의 해(solution)인 eigenfunction U_r로써 표현된 natural motion의 linear combination

$$u(x,t) = \sum_{r=1}^{\infty} U_r(x)\,\eta_r(t)$$

에서 $\eta_r(t)$는 modal coordinate이다.

(a) r번째 mode의 natural frequency를 ω_r이라 할 때, $\eta_r(t)$로 표현되는 modal equation $\ddot{\eta}_r(t) + \omega_r^2\,\eta_r(t) = 0$ 을 유도하시오.

(b) initial displacement $u_0(x)$와 initial velocity $v_0(x)$를 알 때, initial modal displacement $\eta_r(0)$와 initial modal velocity $\dot{\eta}_r(0)$를 유도하시오.

(뒷면에 계속)

4.[6점] 균일한 원형 plate가 $a \le r \le b$인 영역에 놓여 있다. 이 원판의 flexural rigidity는 D이고, 단위면적 당 질량은 ρ이다. $r=a$인 경계는 자유(free)롭고, $r=b$인 경계는 단순지지(simply-support)되어 있다.

(a) plate의 횡 변위(transverse displacement) $w(r,\theta,t)$로써 differential equation of motion과 boundary conditions를 표현하고, 변수분리(separation of variables)를 통하여 공간 좌표 (r,θ)만의 함수인 횡 변위 $W(r,\theta)$로써 differential equation of motion과 boundary condition을 표현하시오.

(b) $W(r,\theta) = R(r)\,\Theta(\theta)$ 와 같은 변수분리를 통하여 일반해(general solution)을 표현하시오 (경계조건 적용하지 않음)

----(참고사항)--

$\tan x = -x \quad \Rightarrow \quad x = 2.03,\ 4.91,\ 7.98,\ \cdots$

$\tan x = -2x \quad \Rightarrow \quad x = 1.84,\ 4.82,\ 7.92,\ \cdots$

$\tan x = \dfrac{1}{2x} \quad \Rightarrow \quad x = 0.65,\ 3.29,\ 6.36,\ \cdots$

$\tan x = \dfrac{1}{x} \quad \Rightarrow \quad x = 0.89,\ 3.43,\ 6.44,\ \cdots$

$\tan x = \dfrac{2}{x} \quad \Rightarrow \quad x = 1.08,\ 3.64,\ 6.58,\ \cdots$

$\tan x = \tanh x \quad \Rightarrow \quad x = 0,\ 3.93,\ 7.07,\ \cdots$

(끝)

연속계진동해석　　중간시험　　해 답　　　　2013년 2학기

[1] free torsional vibration of a uniform circular shaft

(a)

$$\frac{\partial}{\partial x}\left[GJ\frac{\partial\theta(x,t)}{\partial x}\right] = I\frac{\partial^2\theta(x,t)}{\partial t^2} \qquad 0<x<L$$

angular displacement $\theta(x,t)$ is separable in space and time

$$\theta(x,t) = \Theta(x)\cdot\cos(\omega t - \phi)$$

$$-\frac{d}{dx}\left[GJ\frac{d\Theta(x)}{dx}\right] = \omega^2 I\Theta(x) \qquad 0<x<L$$

$$\frac{d^2\Theta(x)}{dx^2} + \beta^2\Theta(x) = 0 \qquad \beta^2 = \omega^2\frac{\rho}{G}$$

boundary conditions

$$\Theta(0) = 0, \qquad \left.\frac{d\Theta}{dx}\right|_{x=L} = 0$$

solution

$$\Theta(x) = A\sin\beta x + B\cos\beta x$$

$$\Theta(0) = B = 0 \quad\Rightarrow\quad \Theta(x) = A\sin\beta x, \qquad \frac{d\Theta(x)}{dx} = \beta A\cos\beta x$$

$$\left.\frac{d\Theta}{dx}\right|_{x=L} = \beta A\cos\beta L = 0 \quad\Rightarrow\quad \cos\beta L = 0$$

$$\beta_r L = \frac{(2r-1)\pi}{2} \qquad r = 1, 2, \cdots$$

$$\omega_r = \frac{(2r-1)\pi}{2}\sqrt{\frac{G}{\rho L^2}}$$: natural frequencies

$$\Theta_r(x) = A_r\sin\frac{(2r-1)\pi}{2}\frac{x}{L}$$: natural modes

(b)

$$\int_0^L \rho\,\Theta_r(x)\Theta_s(x)\,dx \qquad (r \neq s)$$

$$= \rho A_r A_s\int_0^L \sin\frac{(2r-1)\pi x}{2L}\sin\frac{(2s-1)\pi x}{2L}\,dx$$

$$= \rho A_r A_s\int_0^L \frac{1}{2}\left(\cos\frac{r-s}{L}\pi x - \cos\frac{r+s-1}{L}\pi x\right)dx$$

$$= \frac{\rho A_r A_s}{2}\left[\frac{L\pi}{r-s}\sin\frac{r-s}{L}\pi x - \frac{L\pi}{r+s-1}\sin\frac{r+s-1}{L}\pi x\right]_0^L$$

$$= 0$$

The eigenfunctions are orthogonal with respect to the mass density.

[2] (a) free-body diagram for a beam element of length dx

$Q(x,t)$: shearing force $\qquad$ $M(x,t)$: bending moment = $EI\dfrac{\partial^2 y(x,t)}{\partial x^2}$ $\quad\cdots$ ①

force equation of motion (in the vertical direction)

$$\left[Q(x,t) + \frac{\partial Q(x,t)}{\partial x}dx\right] - Q(x,t) = m\,dx\,\frac{\partial^2 y(x,t)}{\partial t^2}$$

expand and cancel appropriate terms

$$\Rightarrow \quad \frac{\partial Q(x,t)}{\partial x} = m\,\frac{\partial^2 y(x,t)}{\partial t^2} \qquad \cdots ②$$

moment equation of motion

$$\left[M(x,t) + \frac{\partial M(x,t)}{\partial x}dx\right] - M(x,t) + \left[Q(x,t) + \frac{\partial Q(x,t)}{\partial x}dx\right]dx = 0$$

cancel appropriate terms

$$\Rightarrow \quad \frac{\partial M(x,t)}{\partial x} + Q(x,t) = 0 \qquad \cdots ③$$

substitute ③ into ②

$$-\frac{\partial^2 M(x,t)}{\partial x^2} = m\,\frac{\partial^2 y(x,t)}{\partial t^2}$$

insert ① into ②

$$\underline{-EI\frac{\partial^4 y(x,t)}{\partial x^4} = m\,\frac{\partial^2 y(x,t)}{\partial t^2}} \qquad 0<x<L$$

substitute ① into ③

$$Q(x,t) = -\frac{\partial M(x,t)}{\partial x} = -EI\frac{\partial^3 y(x,t)}{\partial x^3}$$

$x = 0$에서 $\qquad$ deflection $\quad y(0,t) = 0$

$\qquad\qquad$ bending moment $\quad EI\dfrac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=0} = 0$

$x = L$에서 $\qquad$ bending moment $\quad EI\dfrac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=L} = 0$

$\qquad\qquad$ shearing force $\quad EI\dfrac{\partial^3 y(x,t)}{\partial x^3}\Big|_{x=L} = 0$

(b) separation of variables

$$y(x,t) = Y(x)\cos(\omega t - \phi)$$

differential eigenvalue problem

$$EI\frac{d^4 Y(x)}{dx^4} = \omega^2 m\,Y(x) \qquad 0<x<L$$

$$\Rightarrow \quad \frac{d^4 Y(x)}{dx^4} - \beta^4 Y(x) = 0 \qquad \beta^4 = \frac{\omega^2 m}{EI}$$

$Y(0) = 0 \quad \cdots ④ \qquad\qquad \dfrac{d^2 Y(x)}{dx^2}\Big|_{x=0} = 0 \quad \cdots ⑤$

$\dfrac{d^2 Y(x)}{dx^2}\Big|_{x=L} = 0 \quad \cdots ⑥ \qquad\qquad \dfrac{d^3 Y(x)}{dx^3}\Big|_{x=L} = 0 \quad \cdots ⑦$

general solution

$$Y(x) = A\ \sin\beta x + B\ \cos\beta x + C\ \sinh\beta x + D\ \cosh\beta x$$

$$\frac{d^2 Y(x)}{dx^2} = \beta^2\ (-A\ \sin\beta x - B\ \cos\beta x + C\ \sinh\beta x + D\ \cosh\beta x)$$

④ → $Y(0) = B + D = 0$ ⋯ ⑧

⑤ → $\left.\frac{d^2 Y(x)}{dx^2}\right|_{x=0} = \beta^2\ (-B + D) = 0$ ⋯ ⑨

(i) $\beta = 0$ 일 때

$\beta_0 L = 0$ $\omega_0 = 0$ $Y_0(x) = A_0 x$

(ii) $\beta \neq 0$ 일 때

⑨ → $-B + D = 0$

⑧ ⇒ $B = D = 0,\quad Y(x) = A\ \sin\beta x + C\ \sinh\beta x$

$$\frac{d^2 Y(x)}{dx^2} = \beta^2\ (-A\ \sin\beta x + C\ \sinh\beta x)$$

$$\frac{d^3 Y(x)}{dx^3} = \beta^3\ (-A\ \cos\beta x + C\ \cosh\beta x)$$

⑥ → $\left.\frac{d^2 Y(x)}{dx^2}\right|_{x=L} = \beta^2\ (-A\ \sin\beta L + C\ \sinh\beta L) = 0$

⇒ $-A\ \sin\beta L + C\ \sinh\beta L = 0$ ⋯ ⑩

⑦ → $\left.\frac{d^3 Y(x)}{dx^3}\right|_{x=L} = \beta^3\ (-A\ \cos\beta L + C\ \cosh\beta L) = 0$

⇒ $-A\ \cos\beta L + C\ \cosh\beta L = 0$ ⋯ ⑪

for nontrivial solution of A and C

$$\begin{vmatrix} -\sin\beta L & \sinh\beta L \\ -\cos\beta L & \cosh\beta L \end{vmatrix} = 0$$

$-\sin\beta L\ \cosh\beta L + \cos\beta L\ \sinh\beta L = 0$

⇒ $\tan\beta L - \tanh\beta L = 0$: characteristic equation

(c) $\omega_r = (\beta_r L)^2 \sqrt{\frac{EI}{mL^4}}$

$Y_r(x) = A_r\ [\sinh(\beta_r L)\ \sin(\beta_r L)\frac{x}{L} + \sin(\beta_r L)\ \sinh(\beta_r L)\frac{x}{L}]$ $r = 1, 2, \cdots$

$\beta_0 L = 0$ $\omega_0 = 0$ $Y_0(x) = A_0 x$

$\beta_1 L = 3.93$ $\omega_1 = 15.44\sqrt{\frac{EI}{mL^4}}$

$Y_1(x) = A_1\ (25.4\ \sin 3.93\frac{x}{L} - 0.707\ \sinh 3.93\frac{x}{L})$

$\beta_2 L = 7.07$ $\omega_2 = 50.0\sqrt{\frac{EI}{mL^4}}$

$Y_1(x) = A_1\ (587\ \sin 7.07\frac{x}{L} + 0.707\ \sinh 7.07\frac{x}{L})$

[3] (a) equation of free vibration of a uniform rod

$$EA\,\frac{\partial^2 u(x,t)}{\partial x^2} = m\,\frac{\partial^2 u(x,t)}{\partial t^2} \qquad 0 < x < L \qquad \cdots ①$$

boundary conditions

$$u(0,t) = 0\ , \qquad -EA\,\frac{\partial u(x,t)}{\partial x}\bigg|_{x=L} = 0$$

initial conditions

initial displacement function $\quad u(x,0) = u_0(x)$

initial velocity function $\quad \dfrac{\partial u(x,t)}{\partial t}\bigg|_{t=0} = v_0(x)$

the motion of free vibration caused by initial excitations

$$u(x,t) = \sum_{r=1}^{\infty} u_r(x,t)$$

linear combination of natural motions $u_r(x,t)$

natural motions

$$u_r(x,t) = U_r(x)\,\eta_r(t)$$

$$u(x,t) = \sum_{r=1}^{\infty} U_r(x)\,\eta_r(t) \qquad \cdots ②$$

$U_r(x)$: normal modes , $\qquad \eta_r(t)$: modal coordinates

insert ② into ①

$$EA\sum_{s=1}^{\infty}\frac{d^2U_s(x)}{dx^2}\eta_s(t) = m\sum_{s=1}^{\infty}U_s(x)\frac{d^2\eta_s(t)}{dt^2}$$

$$\Rightarrow \qquad \sum_{s=1}^{\infty}\frac{d}{dx}\left[EA\frac{dU_s(x)}{dx}\right]\eta_s(t) = \sum_{s=1}^{\infty}m\,U_s(x)\frac{d^2\eta_s(t)}{dt^2} \qquad \cdots ③$$

orthonormality relations

$$\int_0^L EA(x)\frac{dU_r(x)}{dx}\,\frac{dU_s(x)}{dx}\,dx = \omega_r^2\,\delta_{rs}$$

$$\int_0^L m(x)\,U_r(x)\,U_s(x)dx = \delta_{rs} \qquad r,s = 1,\ 2,\ \cdots$$

$$\int_0^L U_r(x)\cdot ③\ dx$$

$$\sum_{s=1}^{\infty}\left\{\int_0^L U_r(x)\frac{d}{dx}\left[EA\frac{dU_s(x)}{dx}\right]dx\right\}\eta_s(t) = \sum_{s=1}^{\infty}\left[\int_0^L m\,U_r(x)U_s(x)dx\right]\frac{d^2\eta_s(t)}{dt^2}$$

orthonormality ↓ ↓

$$\sum_{s=1}^{\infty}\left\{\left[U_r(x)EA\frac{dU_s(x)}{dx}\right]\bigg|_{x=L} - \omega_s^2\,\delta_{sr}\right\}\eta_s(t) = \sum_{s=1}^{\infty}\delta_{rs}\,\frac{d^2\eta_s(t)}{dt^2}$$

$$\Rightarrow \quad \sum_{s=1}^{\infty} -\omega_s^2\,\delta_{sr}\,\eta_s(t) = \sum_{s=1}^{\infty}\delta_{rs}\,\frac{d^2\eta_s(t)}{dt^2}$$

$$-\omega_r^2\,\eta_r(t) = \frac{d^2\eta_r(t)}{dt^2}$$

$$\frac{d^2\eta_r(t)}{dt^2} + \omega_r^2 \ \eta_r(t) = 0$$

$$\Rightarrow \quad \ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = 0 \ , \qquad\qquad r = 1, 2, \cdots$$

independent set of modal equations

($\eta_r(t)$: modal coordinates, ω_r : natural frequencies of the rth mode)

(b) solution

$$\eta_r(t) = \eta_r(0) \cos\omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r} \sin\omega_r t$$

initial modal displacements $\eta_r(0)$

$$② \ \& \ u(x,0) = u_0(x) \quad \rightarrow \quad u(x,0) = \sum_{s=1}^{\infty} U_s(x)\eta_s(0) = u_0(x) \qquad \cdots ④$$

$$\int_0^L m\, U_r(x) \cdot ④ \ dx$$

$$\Rightarrow \quad \sum_{s=1}^{\infty}\left[\int_0^L m\, U_r(x)U_s(x)\,dx\right]\eta_s(0) = \int_0^L m\, U_r(x)u_0(x)\,dx$$

orthonormality $\downarrow$

$$\sum_{s=1}^{\infty}\delta_{rs} \ \eta_s(0) = \int_0^L m\, U_r(x)u_0(x)\,dx$$

$$\Rightarrow \quad \eta_r(0) = \int_0^L m\, U_r(x)u_0(x)\,dx \quad , \qquad r = 1, 2, \cdots$$

initial modal velocity $\dot{\eta}_r(0)$

$$② \ \& \ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v_0(x) \quad \rightarrow \quad \dot{u}(x,0) = \sum_{s=1}^{\infty} U_s(x)\,\dot{\eta}_s(0) = v_0(x) \qquad \cdots ⑤$$

$$\int_0^L m(x)U_r(x) \cdot ⑤ \ dx$$

$$\Rightarrow \quad \sum_{s=1}^{\infty}\left[\int_0^L m(x)\,U_r(x)U_s(x)\,dx\right]\dot{\eta}_s(0) = \int_0^L m(x)U_r(x)v_0(x)\,dx$$

orthonormality $\downarrow$

$$\sum_{s=1}^{\infty}\delta_{rs}\,\dot{\eta}_s(0) = \int_0^L m(x)U_r(x)v_0(x)\,dx$$

$$\Rightarrow \quad \dot{\eta}_r(0) = \int_0^L m(x)U_r(x)v_0(x)\,dx \ , \qquad r = 1, 2, \cdots$$

[4] (a) $-D\ \nabla^4 w(r,\theta,t) = \rho \dfrac{\partial^2 w(r,\theta,t)}{\partial t^2}$ $\quad a \le r \le b$

clamped boundary

$$\left.\frac{\partial^2 w}{\partial r^2}\right|_{r=a} = 0 \quad \text{and} \quad \left.\frac{\partial^3 w}{\partial r^3}\right|_{r=a} = 0 \quad \text{along} \quad r = a$$

$$w = 0 \quad \text{and} \quad \left.\frac{\partial^2 w}{\partial r^2}\right|_{r=a} = 0 \quad \text{along} \quad r = b$$

separation of variables $\quad w(r,\theta,t) = W(r,\theta)\ F(t) = W(r,\theta)\ \cos(\omega t - \phi)$

$$\nabla^4 W(r,\theta) - \beta^4\ W(r,\theta) = 0 \qquad \beta^4 = \frac{\omega^2 \rho}{D} \qquad \omega = \beta^2 \sqrt{\frac{D}{\rho}}$$

$$\frac{\partial^2 W}{\partial r^2} = 0 \quad \text{and} \quad \frac{\partial^3 W}{\partial r^3} = 0 \quad \text{along } r = a$$

$$W = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial r^2} = 0 \quad \text{along } r = b$$

(b) $(\nabla^4 - \beta^4)\ W(r,\theta) = 0 \quad \rightarrow \quad (\nabla^2 + \beta^2)(\nabla^2 - \beta^2)\ W(r,\theta) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$(\nabla^2 + \beta^2)\ W_1(r,\theta) = 0 \quad \cdots$ ① $\quad \leftarrow \quad W_1(r,\theta)$

$(\nabla^2 - \beta^2)\ W(r,\theta) = W_1(r,\theta) \qquad W = W_h + W_p = W_2 + W_1$

$(\nabla^2 - \beta^2)\ W_2(r,\theta) = 0 \quad \cdots$ ②

① → same as the equation for a membrane

$$\begin{aligned} W_1(r,\theta) &= (C_{1m} \sin m\theta + C_{2m} \cos m\theta)\,[C_{3m}\ J_m(\beta r) + C_{4m}\ Y_m(\beta r)] \\ &= A_{1m}\ J_m(\beta r) \sin m\theta + A_{2m}\ J_m(\beta r) \cos m\theta \\ &\quad + A_{3m}\ Y_m(\beta r) \sin m\theta + A_{4m}\ Y_m(\beta r) \cos m\theta \end{aligned}$$

② $\dfrac{\partial^2 W_2(r,\theta)}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial W_2(r,\theta)}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 W_2(r,\theta)}{\partial \theta^2} - \beta^2\ W_2(r,\theta) = 0$

$$W_2(r,\theta) = R(r)\ \Theta(\theta)$$

$$\frac{r^2}{R(r)}\frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + \frac{1}{\Theta(\theta)}\frac{d^2\Theta(\theta)}{d\theta^2} - r^2\beta^2 = 0$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad -m^2$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + m^2 \Theta(\theta) = 0 \qquad \frac{d^2 R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - (\beta^2 + \frac{m^2}{r^2}) R(r) = 0$$

$$\Theta_m(\theta) = C_{1m} \sin m\theta + C_{2m} \cos m\theta \qquad R_m(y) = C_{3m}\ I_m(\beta r) + C_{4m}\ K_m(\beta r)$$

$$m = 0,\ 1,\ 2,\ \cdots$$

$$\begin{aligned} W_2(r,\theta) &= (C_{1m} \sin m\theta + C_{2m} \cos m\theta)\ [C_{3m}\ I_m(\beta r) + C_{4m}\ K_m(\beta r)] \\ &= B_{1m}\ I_m(\beta r) \sin m\theta + B_{2m}\ I_m(\beta r) \cos m\theta \\ &\quad + B_{3m}\ K_m(\beta r) \sin m\theta + B_{4m} K_m(\beta r) \cos m\theta \end{aligned}$$

general solution

$$\begin{aligned} W_{mn}(r,\theta) &= W_1(r,\theta) + W_2(r,\theta) \\ &= [A_{1m}\ J_m(\beta r) + A_{3m}\ Y_m(\beta r) + B_{1m}\ I_m(\beta r) + B_{3m}\ K_m(\beta r)] \sin m\theta \\ &\quad + [A_{2m}\ J_m(\beta r) + A_{4m}\ Y_m(\beta r) + B_{2m}\ I_m(\beta r) + B_{4m}\ K_m(\beta r)] \cos m\theta \end{aligned}$$