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[30 점]

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1.[8점] 균일한 보(beam)의 굽힘 진동(bending vibration)을 횡방향 변위(transverse displacement) $y(x,t)$로 표현한다. 이 막대는 길이가 L이고 전체 질량이 M이며, 굽힘 강성(flexural rigidity)이 EI이고, 외력은 작용하지 않는다. x=0인 지점에서 자유롭고 x=L인 지점에서 고정(clamp)되어 있는 외팔보(cantilever)이다.

(a) Newton의 운동 제2법칙에 근거하여 differential equation of motion을 유도하고, 경계조건을 제시하시오.

(b) 변수분리(separation of variation)법을 적용하여 특성방정식(characteristic equation)을 구하시오.

2. [8점] 균일한 막대(rod)의 종방향 자유진동을 고려한다. 이 막대는 길이가 L이고, 단위길이당 질량이 m이며, 종강성(axial rigidity)이 EA이다. 이 막대는 x=0인 지점에서 고정되어 있고, x=L인 지점에 집중질량 M이 달려 있다. 변수분리(separation of variable)된 eigenvalue problem은 다음과 같다.

$$-EA\frac{d^2U(x)}{dx^2} = \omega^2 m(x)U(x)\ ,\quad U(0) = 0\ ,\quad EA\frac{dU(x)}{dx}\bigg|_{x=L} = \omega^2 M\,U(L)$$

(a) natural mode의 orthogonality를 나타내는 식을 유도하시오.

(b) 정규화(normalization) 된 eigenfunction $U_r(x)$의 무한급수로 표현된 막대의 변위(displacement)는 $U(x) = \sum_{r=1}^{\infty} c_r\, U_r(x)$ 이다. 계수 c_r을 나타내는 표현을 유도하시오.

3. [8점] 균일한 현(string)의 횡방향 자유진동을 고려한다. 이 현은 길이가 L이고, 단위길이당 질량이 ρ이며, 장력이 T이다. 양쪽 끝에서 고정되어 있다. 횡방향 변위 $y(x,t)$로 표현된 운동방정식이 다음과 같다.

$$T\frac{\partial^2 y(x,t)}{\partial x^2} = \rho\frac{\partial^2 y(x,t)}{\partial t^2}$$

eigenvalue problem의 해(solution)인 eigenfunction Y_r로써 표현된 natural motion

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$$

에서 $\eta_r(t)$는 modal coordinate이다.

(a) r번째 mode의 natural frequency를 ω_r이라 할 때, $\eta_r(t)$로 표현되는 modal equation $\ddot{\eta}_r(t) + \omega_r^2\,\eta_r(t) = 0$ 을 유도하시오.

(b) initial displacement $y_0(x)$와 initial velocity $v_0(x)$를 알 때, initial modal displacement $\eta_r(0)$와 initial modal velocity $\dot{\eta}_r(0)$를 유도하시오.

(뒷면에 계속)

4. [6점] 균일한 사각 plate가 $0 \le x \le a$, $0 \le y \le b$인 영역에 놓여 있다. 이 사각 판의 flexural rigidity는 D이고, 단위면적 당 질량은 ρ이다. $x=0$과 $x=a$인 edge는 단순지지(simply-support)되어 있고, 그 외 부분은 자유롭다.

(a) plate의 횡 변위(transverse displacement) $w(x,y,t)$로써 differential equation of motion과 boundary condition을 표현하시오.

(b) 변수분리(separation of variables)를 통하여 공간 좌표 (x,y)만의 함수인 횡 변위 $W(x,y)$로써 differential equation of motion과 boundary condition을 표현하시오.

(끝)

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[1] 단위길이당 질량 = M/L

(a) free-body diagram for a beam element of length dx

$Q(x,t)$: shearing force　　　$M(x,t)$: bending moment = $EI\dfrac{\partial^2 y(x,t)}{\partial x^2}$　　… ①

force equation of motion (in the vertical direction)

$$\left[Q(x,t) + \frac{\partial Q(x,t)}{\partial x}dx\right] - Q(x,t) = \frac{M}{L}\,dx\,\frac{\partial^2 y(x,t)}{\partial t^2}$$

expand and cancel appropriate terms

$$\Rightarrow \quad \frac{\partial Q(x,t)}{\partial x} = \frac{M}{L}\frac{\partial^2 y(x,t)}{\partial t^2} \qquad \cdots ②$$

moment equation of motion

$$\left[M(x,t) + \frac{\partial M(x,t)}{\partial x}dx\right] - M(x,t) + \left[Q(x,t) + \frac{\partial Q(x,t)}{\partial x}dx\right]dx = 0$$

cancel appropriate terms

$$\Rightarrow \quad \frac{\partial M(x,t)}{\partial x} + Q(x,t) = 0 \qquad \cdots ③$$

substitute ③ into ②

$$-\frac{\partial^2 M(x,t)}{\partial x^2} = \frac{M}{L}\frac{\partial^2 y(x,t)}{\partial t^2}$$

insert ① into ②

$$\underline{-EI\frac{\partial^4 y(x,t)}{\partial x^4} = \frac{M}{L}\frac{\partial^2 y(x,t)}{\partial t^2}} \qquad 0 < x < L$$

substitute ① into ③

$$Q(x,t) = -\frac{\partial M(x,t)}{\partial x} = -EI\frac{\partial^3 y(x,t)}{\partial x^3}$$

$x = 0$에서　　bending moment $EI\dfrac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=0} = 0$

　　　　　　shearing force　$EI\dfrac{\partial^3 y(x,t)}{\partial x^3}\Big|_{x=0} = 0$

$x = L$에서　　deflection $y(L,t) = 0$

　　　　　　slope $\dfrac{\partial y(x,t)}{\partial x}\Big|_{x=L} = 0$

(b) separation of variables

$$y(x,t) = Y(x)\cos(\omega t - \phi)$$

differential eigenvalue problem

$$EI\frac{d^4Y(x)}{dx^4} = \omega^2\frac{M}{L}Y(x) \qquad 0 < x < L$$

$$\Rightarrow \quad \frac{d^4Y(x)}{dx^4} - \beta^4 Y(x) = 0 \qquad \beta^4 = \frac{\omega^2 M}{EIL}$$

$$\left.\frac{d^2Y(x)}{dx^2}\right|_{x=0} = 0 \qquad \left.\frac{d^3Y(x)}{dx^3}\right|_{x=0} = 0$$

$$Y(L) = 0 \qquad \left.\frac{dY(x)}{dx}\right|_{x=L} = 0$$

general solution

$$Y(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

$$\frac{d^2Y(x)}{dx^2} = \beta^2(-A\sin\beta x - B\cos\beta x + C\sinh\beta x + D\cosh\beta x)$$

$$\left.\frac{d^2Y(x)}{dx^2}\right|_{x=0} = \beta^2(-B + D) = 0 \qquad \Rightarrow \qquad D = B$$

$$\frac{d^3Y(x)}{dx^3} = \beta^3(-A\cos\beta x + B\sin\beta x + C\cosh\beta x + B\sinh\beta x)$$

$$\left.\frac{d^3Y(x)}{dx^3}\right|_{x=0} = \beta^3[-A + C] = 0 \qquad \Rightarrow \qquad C = A$$

$$Y(x) = A(\sin\beta x + \sinh\beta x) + B(\cos\beta x + \cosh\beta x)$$

$$Y(L) = 0$$

$$\Rightarrow \qquad A(\sin\beta L + \sinh\beta L) + B(\cos\beta L + \cosh\beta L) = 0 \qquad \cdots ④$$

$$\frac{dY(x)}{dx} = \beta[A(\cos\beta x + \cosh\beta x) + B(-\sin\beta x + \cosh\beta x)]$$

$$\left.\frac{dY(x)}{dx}\right|_{x=L} = 0$$

$$\Rightarrow \qquad \beta[A(\cos\beta L + \cosh\beta L) + B(-\sin\beta L + \sinh\beta L)] = 0 \cdots ⑤$$

for nontrivial solution of A and B

$$\begin{vmatrix} (\sin\beta L + \sinh\beta L) & (\cos\beta L + \cosh\beta L) \\ (\cos\beta L + \cosh\beta L) & (-\sin\beta L + \sinh\beta L) \end{vmatrix} = 0$$

$$-\sin^2\beta L + \sinh^2\beta L - \cos^2\beta L - 2\cos\beta L\cosh\beta L - \cosh^2\beta L = 0$$

$$\Rightarrow \qquad \cos\beta L\cosh\beta L = -1 \qquad \text{: characteristic equation}$$

[2] (a) natural frequencies ω_r , ω_s

natural modes $U_r(x)$, $U_s(x)$

equations

$$-EA\frac{d^2U_r(x)}{dx^2} = \omega_r^2\, m(x)\, U_r(x) \qquad 0 < x < L$$

$$-EA\frac{d^2U_s(x)}{dx^2} = \omega_s^2\, m(x)\, U_s(x) \qquad 0 < x < L$$

r번째 mode

$$\omega_r^2\, m\int_0^L U_s(x)U_r(x)dx = -EA\int_0^L U_s(x)\frac{d^2U_r(x)}{dx^2}dx$$

$$= -EA\, U_s(x)\frac{dU_r(x)}{dx}\bigg|_0^L + EA\int_0^L \frac{dU_s(x)}{dx}\frac{dU_r(x)}{dx}dx$$

$$= -\omega_r^2\, M\, U_s(L)\, U_r(L) + EA\int_0^L \frac{dU_s(x)}{dx}\frac{dU_r(x)}{dx}dx$$

$$\Rightarrow \quad EA\int_0^L \frac{dU_s(x)}{dx}\frac{dU_r(x)}{dx}dx = \omega_r^2\left[m\int_0^L U_s(x)U_r(x)dx + M\, U_r(L)\, U_s(L)\right]$$

s번째 mode

$$\omega_s^2\, m\int_0^L U_r(x)U_s(x)dx = -EA\int_0^L U_r(x)\frac{d^2U_s(x)}{dx^2}$$

$$= -EA\, U_r(x)\frac{dU_s(x)}{dx}\bigg|_0^L + EA\int_0^L \frac{dU_r(x)}{dx}\frac{dU_s(x)}{dx}dx$$

$$= -\omega_s^2\, M\, U_r(L)\, U_s(L) + EA\int_0^L \frac{dU_r(x)}{dx}\frac{dU_s(x)}{dx}dx$$

$$\Rightarrow \quad EA\int_0^L \frac{dU_r(x)}{dx}\frac{dU_s(x)}{dx}dx = \omega_s^2\left[m\int_0^L U_r(x)U_s(x)dx + M\, U_r(L)\, U_s(L)\right]$$

$$(\omega_r^2-\omega_s^2)\left[\int_0^L m(x)\, U_r(x)U_s(x)dx + M\, U_r(L)\, U_s(L)\right] = 0$$

$$r \neq s,\ \omega_r \neq \omega_s \quad \Rightarrow \quad \underline{m\int_0^L U_r(x)U_s(x)dx + M\, U_r(L)\, U_s(L) = 0}$$

The eigenfunctions $U_r(x)$ and $U_s(x)$ are orthogonal w.r.t. the mass density m and the lumped mass M at $x = L$.

$$\Rightarrow \qquad \underline{EA\int_0^L \frac{dU_s(x)}{dx}\frac{dU_r(x)}{dx}dx = 0} \qquad r \neq s,\ \omega_r \neq \omega_s$$

The derivatives of the eigenfunctions are orthogonal w.r.t. the axial stiffness $EA(x)$.

(b) normalization

for $r = s$ $r = 1, 2, \cdots$

$$\int_0^L m(x)\,[U_r(x)]^2\,dx + M\,[U_r(L)]^2 = 1$$

$$\int_0^L EA(x)\left[\frac{dU_r(x)}{dx}\right]^2 dx = \omega_r^2\left[\int_0^L m(x)\,[U_r(x)]^2\,dx + M\,[U_r(L)]^2\right] = \omega_r^2$$

orthonormality relations

$$\int_0^L m(x)\,U_r(x)\,U_s(x)dx + M\,U_r(L)\,U_s(L) = \delta_{rs}$$

$$\Rightarrow \quad \int_0^L m(x)\,U_r(x)\,U_s(x)dx = \delta_{rs} - M\,U_r(L)\,U_s(L)$$

$$\int_0^L EA(x)\,\frac{dU_r(x)}{dx}\,\frac{dU_s(x)}{dx}\,dx = \omega_r^2\,\delta_{rs}$$

expansion theorem

$$U(x) = \sum_{s=1}^{\infty} c_s\,U_s(x)$$

$$\int_0^L m(x)\,U_r(x)\cdot(\text{equation})\,dx$$

$$\Rightarrow \quad \int_0^L m(x)\,U_r(x)\,U(x)\,dx = \int_0^L m(x)\,U_r(x)\left[\sum_{s=1}^{\infty} c_s\,U_s(x)\right]dx$$

$$= \sum_{s=1}^{\infty} c_s \int_0^L m(x)\,U_r(x)\,U_s(x)\,dx = \sum_{s=1}^{\infty} c_s\,[\delta_{rs} - M\,U_r(L)\,U_s(L)]$$

$$= c_r - M\,U_r(L)\sum_{s=1}^{\infty} c_s\,U_s(L) = c_r - M\,U_r(L)\,U(L)$$

$$\therefore \; c_r = \int_0^L m(x)\,U_r(x)\,U(x)\,dx + M\,U_r(L)\,U(L)$$

[3] (a) $\frac{\partial}{\partial x}\left[T\frac{\partial y(x,t)}{\partial x}\right] = \rho\frac{\partial^2 y(x,t)}{\partial t^2}$ $\qquad 0 < x < L \qquad \cdots$ ①

boundary conditions

$y(0,t) = 0, \qquad y(L,t) = 0$

initial conditions

initial displacement function $\quad y(x,0) = y_0(x)$

initial velocity function $\quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = v_0(x)$

the motion of free vibration caused by initial excitations

$$y(x,t) = \sum_{r=1}^{\infty} y_r(x,t)$$

linear combination of natural motions $y_r(x,t)$

natural motions

$$y_r(x,t) = Y_r(x)\,\eta_r(t)$$

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t) \qquad \cdots ②$$

insert ② into ①

$$\frac{\partial}{\partial x}\left[T\sum_{s=1}^{\infty}\frac{dY_s(x)}{dx}\eta_s(t)\right] = \rho\sum_{s=1}^{\infty}Y_s(x)\frac{d^2\eta_s(t)}{dt^2}$$

$$\Rightarrow \qquad \sum_{s=1}^{\infty}\frac{d}{dx}\left[T\frac{dY_s(x)}{dx}\right]\eta_s(t) = \sum_{s=1}^{\infty}\rho\,Y_s(x)\frac{d^2\eta_s(t)}{dt^2} \qquad \cdots ③$$

$\int_0^L Y_r(x)\cdot ③\; dx$

$$\sum_{s=1}^{\infty}\left\{\int_0^L Y_r(x)\frac{d}{dx}\left[T\frac{dY_s(x)}{dx}\right]dx\right\}\eta_s(t) = \sum_{s=1}^{\infty}\left[\int_0^L \rho\,Y_r(x)\,Y_s(x)dx\right]\frac{d^2\eta_s(t)}{dt^2}$$

orthonormality $\downarrow \qquad\qquad\qquad \downarrow$

$$\sum_{s=1}^{\infty}\left\{-\omega_s^2\,\delta_{sr}\right\}\quad \eta_s(t) \qquad = \qquad \sum_{s=1}^{\infty}\left[\delta_{rs}\right]\quad \frac{d^2\eta_s(t)}{dt^2}$$

$$-\omega_r^2 \quad \eta_r(t) \qquad = \qquad 1 \quad \frac{d^2\eta_r(t)}{dt^2}$$

$\Rightarrow \quad \ddot{\eta}_r(t) + \omega_r^2\,\eta_r(t) = 0\ , \qquad r = 1,\ 2,\ \cdots$

independent set of modal equations

($\eta_r(t)$: modal coordinates, $\quad \omega_r$: natural frequencies of the rth mode)

(b)

initial modal displacements $\eta_r(0)$

② & $y(x,0) = y_0(x)$ → $y(x,0) = \sum_{s=1}^{\infty} Y_s(x)\,\eta_s(0) = y_0(x)$ ⋯ ④

$$\int_0^L \rho(x)\,Y_r(x) \cdot ④ \; dx$$

$$\Rightarrow \quad \sum_{s=1}^{\infty}\left[\int_0^L \rho\, Y_r(x)\,Y_s(x)\,dx\right]\eta_s(0) = \int_0^L \rho\, Y_r(x)\,y_0(x)\,dx$$

orthonormality ↓

$$\sum_{s=1}^{\infty}[\delta_{rs}] \quad \eta_s(0) = \int_0^L \rho\, Y_r(x)\,y_0(x)\,dx$$

$$\Rightarrow \quad \eta_r(0) = \int_0^L \rho\, Y_r(x)\,y_0(x)\,dx \;, \qquad r = 1,\ 2,\ \cdots$$

initial modal velocity $\dot{\eta}_r(0)$

② & $\left.\dfrac{\partial y(x,t)}{\partial t}\right|_{t=0} = v_0(x)$ → $\dot{y}(x,0) = \sum_{s=1}^{\infty} Y_s(x)\,\dot{\eta}_s(0) = v_0(x)$ ⋯ ⑤

$$\int_0^L \rho\, Y_r(x) \cdot ⑤ \; dx$$

$$\Rightarrow \quad \sum_{s=1}^{\infty}\left[\int_0^L \rho\, Y_r(x)\,Y_s(x)\,dx\right]\dot{\eta}_s(0) = \int_0^L \rho\, Y_r(x)\,v_0(x)\,dx$$

orthonormality ↓

$$\sum_{s=1}^{\infty}[\delta_{rs}] \quad \dot{\eta}_s(0) = \int_0^L \rho\, Y_r(x)\,v_0(x)\,dx$$

$$\Rightarrow \quad \dot{\eta}_r(0) = \int_0^L \rho\, Y_r(x)\,v_0(x)\,dx \;, \qquad r = 1,\ 2,\ \cdots$$

[4] (a) $-D\ \nabla^4 w(\mathbf{r},t) \;=\; \rho \dfrac{\partial^2 w(\mathbf{r},t)}{\partial t^2}$ $\qquad 0 \le x \le a,\ 0 \le y \le b$

simply-supported boundaries

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{along } x = 0,\ a$$

free boundaries

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^3 w}{\partial x^3} = 0 \qquad \text{along } y = 0,\ b$$

(b) separation of variables

$$w(x,y,t) = W(x,y)\ F(t) = W(x,y)\ \cos(\omega t - \phi)$$

$$\nabla^4 W(x,y) - \beta^4\ W(x,y) = 0 \qquad \beta^4 = \frac{\omega^2 \rho}{D}$$

$$W = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial x^2} = 0 \qquad \text{along } x = 0,\ a$$

$$\frac{\partial^2 W}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^3 W}{\partial y^3} = 0 \qquad \text{along } y = 0,\ b$$