기 계 진 동 학 (제4판) 연습문제 풀이 제1장 (2023년 과제)

[1.1절]

1.30 ω_n = 3 rad/s, x_0 = 1.2 mm, v_0 = 2.34 mm/s, $x(t)$ = ? Compute and plot.

$x(t) = A\ \sin(\omega_n t + \phi)$ $\dot{x}(t) = \omega_n\ A\ \cos(\omega_n t + \phi)$

$x(0) = A\ \sin\phi = x_0 = 1.2\ \text{mm}\ >0$ ⋯ ①

$\dot{x}(0) = \omega_n\ A\ \cos\phi = v_0 \quad \Rightarrow \quad A\ \cos\phi = \dfrac{v_0}{\omega_n} = \dfrac{2.34\ \text{mm/s}}{3\ \text{rad/s}} = 0.78\ \text{mm}\ >0$ ⋯ ②

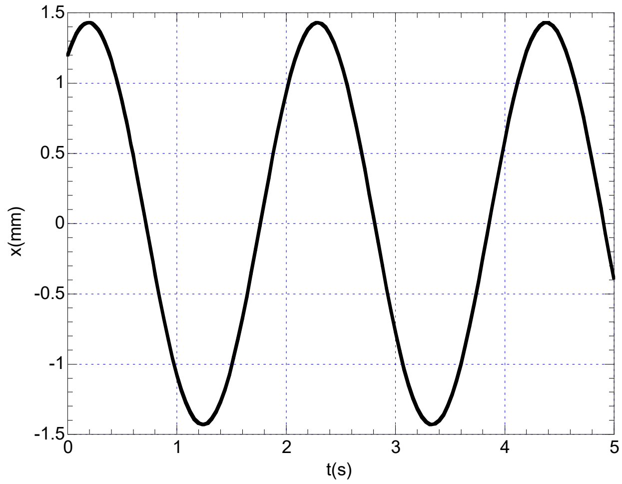
$\sin\phi > 0$, $\cos\phi > 0$ 이므로, $0<\phi<\dfrac{\pi}{2}$

①^2 + ②^2 $\Rightarrow$ $A = \sqrt{(1.2\ \text{mm})^2 + (0.78\ \text{mm})^2} = 1.431\ \text{mm}$

① ÷ ② $\Rightarrow$ $\phi = \tan^{-1}\dfrac{(1.2\ \text{mm})}{(0.78\ \text{mm})} = \tan^{-1}1.538 = 0.994\ \text{rad}\ (= 57.0°)$

$\therefore\ x(t) = 1.431\ \sin(3.00\ t + 0.994)$ mm

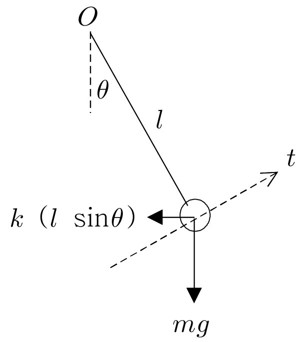
Plot



[1.2절]

1.21 [뉴튼 법칙 사용]

<방법 1: 오일러 법칙>



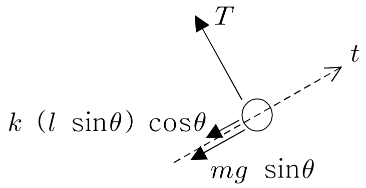
$\Sigma M_O = J\ddot{\theta}$ (질량관성모멘트 $J = m\,l^2$)

$\Rightarrow\ -m\,g\,(l\ \sin\theta) - k\,(l\ \sin\theta)\,(l\ \cos\theta) = (m\,l^2)\,\ddot{\theta}$

$\Rightarrow\ m\,l^2\,\ddot{\theta} + (m\,g + k\,l\,\cos\theta)\,l\,\sin\theta = 0$, $l \neq 0$,

$\Rightarrow\ m\,l\,\ddot{\theta} + (m\,g + k\,l\,\cos\theta)\,\sin\theta = 0$

<방법 2>



$\Sigma F_t = m\,a_t$ (가속도 $a_t = l\,\ddot{\theta}$)

$\Rightarrow\ -m\,g\,\sin\theta - k\,(l\ \sin\theta)\,\cos\theta = m\,(l\,\ddot{\theta})$

$\Rightarrow\ m\,l\,\ddot{\theta} + (m\,g + k\,l\,\cos\theta)\,\sin\theta = 0$

$\theta \simeq 0$ 이면 $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$

$\Rightarrow\ m\,l\,\ddot{\theta} + (m\,g + k\,l)\,\theta = 0$

$\Rightarrow\ \ddot{\theta} + (\frac{g}{l} + \frac{k}{m})\,\theta = 0$ 고유진동수 $\omega_n = \sqrt{\frac{g}{l} + \frac{k}{m}}$

[에너지방법]

운동에너지 $T = \frac{1}{2}m\,v^2 = \frac{1}{2}m\,(l\dot{\theta})^2 = \frac{1}{2}m\,l^2\,\dot{\theta}^2$

위치에너지 $U = mgl\,(1 - \cos\theta) + \frac{1}{2}k\,(l\,\theta)^2$

$\frac{d}{dt}(T + U) = \frac{d}{dt}[\frac{1}{2}m\,l^2\,\dot{\theta}^2 + mgl\,(1 - \cos\theta) + \frac{1}{2}k\,l^2\,\theta^2] = 0$

$\Rightarrow\ m\,l^2\,\dot{\theta}\,\ddot{\theta} + mgl\,\sin\theta\,\dot{\theta} + k\,l^2\,\theta\,\dot{\theta} = 0,$

$\theta \simeq 0$ 이면 $\sin\theta \simeq \theta$

$[m\,l\,(l\,\ddot{\theta} + g\,\theta) + k\,l^2\,\theta]\,\dot{\theta} = 0 \quad \Rightarrow\ m\,l\,\ddot{\theta} + (m\,g + k\,l)\,\theta = 0$

$\Rightarrow\ \ddot{\theta} + (\frac{g}{l} + \frac{k}{m})\,\theta = 0$

고유진동수 $\omega_n = \sqrt{\frac{g}{l} + \frac{k}{m}}$

[1.3절]

1.54 ω_n = 2 rad/s, ζ = 0.1, v_0 = 0이고, 초기변위 x_0 = 10, 100 mm에 대해 $x(t)$ plot.

$$x(t) = A\ e^{-\zeta\omega_n t}\ \sin(\omega_d t + \phi), \quad \dot{x}(t) = A\ e^{-\zeta\omega_n t}\ [-\zeta\omega_n \sin(\omega_d t + \phi) + \omega_d \cos(\omega_d t + \phi)]$$

$$x(0) = A\ \sin\phi = x_0 \quad > 0 \quad \cdots ①$$

$$\dot{x}(0) = A\ (-\zeta\omega_n\ \sin\phi + \omega_d\ \cos\phi) = v_0 = 0$$

$$\Rightarrow \quad A\ \cos\phi = \frac{\zeta\omega_n x_0}{\omega_d} = \frac{\zeta}{\sqrt{1-\zeta^2}}\,x_0 = \frac{0.1}{\sqrt{1-0.1^2}}\,x_0 = 0.1005\ x_0 > 0 \quad \cdots ②$$

$$①^2 + ②^2 \quad \Rightarrow \quad A = \sqrt{1^2 + 0.1005^2}\,x_0 = 1.005\ x_0$$

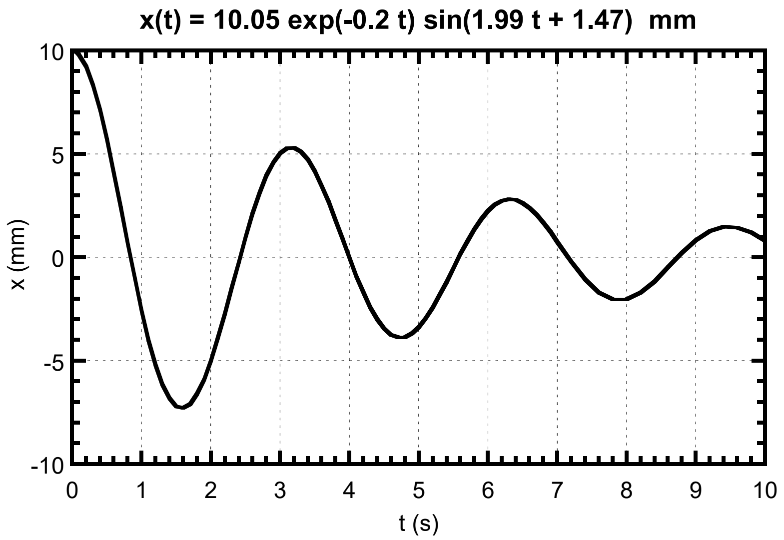
$$① \div ② \quad \Rightarrow \quad \phi = \tan^{-1}\frac{1}{0.1005} = \tan^{-1}(9.95) = 84.2° = 1.471\ \text{rad}$$

$$\zeta\omega_n = (0.1)\ (2\ \text{rad/s}) = 0.2\ \text{rad/s}, \quad \omega_d = \sqrt{1-0.1^2}\,(2\ \text{rad/s}) = 1.990\ \text{rad/s}$$

$$x(t) = 1.005\ x_0\ e^{-0.2t}\ \sin(1.990\,t + 1.471)$$

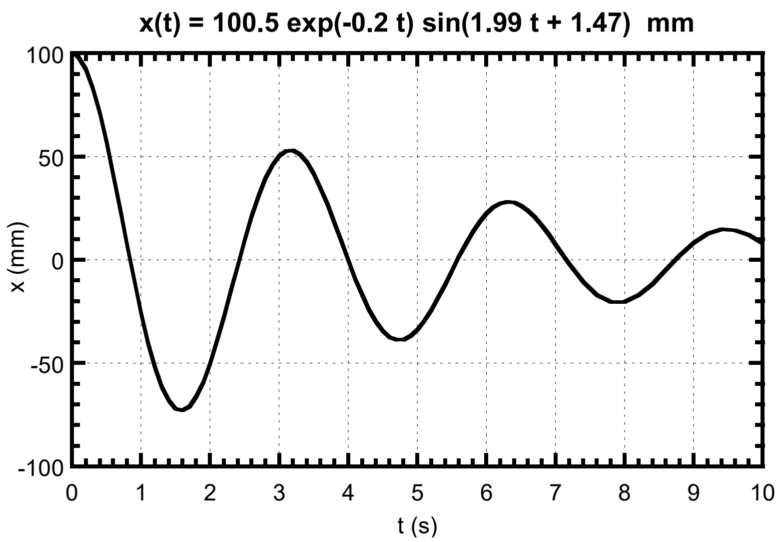
x_0 = 10 mm일 때, $x(t) = 1.005\ (10\ \text{mm})\ e^{-0.2t}\ \sin(1.990\,t + 1.471)$

$= 10.05\ e^{-0.2t}\ \sin(1.990\,t + 1.471)$ mm



x_0 = 100 mm일 때, $x(t) = 1.005\ (100\ \text{mm})\ e^{-0.2t}\ \sin(1.990\,t + 1.471)$

$= 100.5\ e^{-0.2t}\ \sin(1.990\,t + 1.471)$ mm



[1.4절]

1.80 statics $J = \frac{1}{2}mr^2$, kinematics $x = r\theta$, $\dot{x} = r\dot{\theta}$

spring $\Delta l = (r+a)\theta$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}(\frac{1}{2}mr^2)\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2$$

$$U = 2\ [\frac{1}{2}k\ (\Delta l)^2\ ] = k\,[(r+a)\theta]^2 = k\,(r+a)^2\theta^2$$

$$\frac{d}{dt}(T+U) = \frac{d}{dt}[\frac{3}{4}mr^2\dot{\theta}^2 + k\,(r+a)^2\theta^2]$$

$$= [\frac{3}{2}mr^2\ddot{\theta} + 2k\,(r+a)^2\theta]\ \dot{\theta} = 0\ , \quad \dot{\theta} \neq 0$$

equation of motion $\frac{3}{2}mr^2\ddot{\theta} + 2k\,(r+a)^2\theta = 0$,

natural frequency $\omega_n = \sqrt{\frac{4k(a+r)^2}{3mr^2}} = 2\frac{a+r}{r}\sqrt{\frac{k}{3m}}$

[1.5절]

1.84 m = 1,200 kg, l = 0.20 m, b = 0.12 m, J = 12 kg · m^2,

steel E = 200 GPa = 200×10^9 N/m^2 , G = 80 GPa = 80×10^9 N/m^2

$A = b^2$ = (0.12 m)^2 = 0.0144 m^2

$J_p = \frac{1}{6}b^4 = \frac{1}{6}$ (0.12 m)^4 = 34.56×10^{-6} m^4

longitudinal vibration

$$k = \frac{EA}{l} = \frac{(200\times10^9\ \mathrm{N/m^2})(0.0144\ \mathrm{m^2})}{0.20\ \mathrm{m}} = 14.40\times10^9\ \mathrm{N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{14.40\times10^9\ \mathrm{N/m}}{1{,}200\ \mathrm{kg}}} = 3{,}464\ \mathrm{rad/s} \quad \Rightarrow \quad (\omega_n)_{longitudinal} = 3{,}460\ \mathrm{rad/s}$$

torsional vibration

$$k_t = \frac{GJ_p}{l} = \frac{(80\times10^9\ \mathrm{N/m^2})(34.56\times10^{-6}\ \mathrm{m^4})}{0.20\ \mathrm{m}} = 13.824\times10^6\ \mathrm{N\cdot m/rad}$$

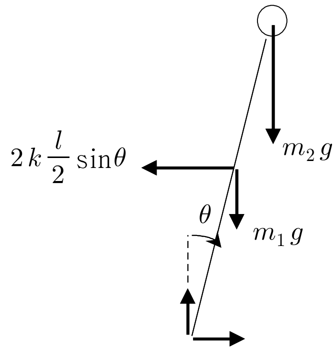
$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{13.824\times10^6\ \mathrm{N\cdot m/rad}}{12\ \mathrm{kg\cdot m^2}}} = 1{,}073.3\ \mathrm{rad/s} \quad \Rightarrow \quad (\omega_n) = 1{,}073\ \mathrm{rad/s}$$

Longitudinal frequency is larger.

[1.8절]

1.113 <방법 1> Euler's 2nd law

F.B.D.



$$\Sigma M_O = J\ddot{\theta}$$

$$m_1 g\left(\frac{l}{2}\sin\theta\right) + m_2 g\,(l\sin\theta) - 2k\frac{l}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) = \left(\frac{1}{3}m_1 l^2 + m_2 l^2\right)\ddot{\theta}$$

$$\Rightarrow \left(\frac{1}{2}m_1 + m_2\right) g\, l\sin\theta - \frac{1}{2}k\,l^2\sin\theta\cos\theta = \left(\frac{1}{3}m_1 + m_2\right) l^2\ddot{\theta}$$

$\theta \approx 0$ 일 때, $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2} \approx 1$

$$\Rightarrow \left(\frac{1}{2}m_1 + m_2\right) g\, l\,\theta - \frac{1}{2}k\,l^2\,\theta = \left(\frac{1}{3}m_1 + m_2\right) l^2\ddot{\theta}$$

$$\Rightarrow \left(\frac{1}{3}m_1 + m_2\right) l\,\ddot{\theta} + \left[\frac{1}{2}k\,l - \left(\frac{1}{2}m_1 + m_2\right) g\right]\theta = 0$$

<방법 2> energy method

$$T = \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}J_2\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}m_1 l^2\right)\dot{\theta}^2 + \frac{1}{2}\left(m_2 l^2\right)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}m_1 + m_2\right) l^2\dot{\theta}^2$$

$$U = 2\left[\frac{1}{2}k\left(\frac{l}{2}\theta\right)^2\right] - m_1 g\frac{l}{2}(1 - \cos\theta) - m_2 g\, l\,(1 - \cos\theta)$$

$$= \frac{1}{4}k\,l^2\,\theta^2 - \left(\frac{1}{2}m_1 + m_2\right) g\, l\,(1 - \cos\theta)$$

$$\frac{d}{dt}(T + U) = \frac{d}{dt}\left[\frac{1}{2}\left(\frac{1}{3}m_1 + m_2\right) l^2\dot{\theta}^2 + \frac{1}{4}k\,l^2\,\theta^2 - \left(\frac{1}{2}m_1 + m_2\right) g\, l\,(1 - \cos\theta)\right] = 0$$

$$\Rightarrow \left(\frac{1}{3}m_1 + m_2\right) l^2\,\dot{\theta}\,\ddot{\theta} + \frac{1}{2}k\,l^2\,\theta\,\dot{\theta} - \left(\frac{1}{2}m_1 + m_2\right) g\, l\sin\theta\,\dot{\theta} = 0$$

$$\Rightarrow \left(\frac{1}{3}m_1 + m_2\right) l^2\,\ddot{\theta} + \frac{1}{2}k\,l^2\,\theta - \left(\frac{1}{2}m_1 + m_2\right) g\, l\sin\theta = 0$$

$\theta \approx 0$ 이면, $\sin\theta \approx \theta$

$$\Rightarrow \left(\frac{1}{3}m_1 + m_2\right) l^2\,\ddot{\theta} + \frac{1}{2}k\,l^2\,\theta - \left(\frac{1}{2}m_1 + m_2\right) g\, l\,\theta = 0$$

$$\Rightarrow \left(\frac{1}{3}m_1 + m_2\right) l\,\ddot{\theta} + \left[\frac{1}{2}k\,l - \left(\frac{1}{2}m_1 + m_2\right) g\right]\theta = 0$$

안정성 검토

$\frac{1}{2}k\,l - \left(\frac{1}{2}m_1 + m_2\right) g > 0$ 이면,

$$\ddot{\theta} + \omega_n^2\,\theta = 0 \qquad \omega_n = \sqrt{\frac{\frac{1}{2}k\,l - \left(\frac{1}{2}m_1 + m_2\right) g}{\left(\frac{1}{3}m_1 + m_2\right) l}}$$

$\theta(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t} = A_1 \sin\omega_n t + A_2 \cos\omega_n t$ 　 진동 ⇒ 안정(stable)

$\frac{1}{2}k\,l - \left(\frac{1}{2}m_1 + m_2\right) g = 0$ 이면,

$\ddot{\theta} = 0 \Rightarrow \theta(t) = a_1 + a_2 t$ 　 비진동 증폭 ⇒ 불안정(unstable)

$\frac{1}{2}k\,l - \left(\frac{1}{2}m_1 + m_2\right)g < 0$ 이면,

$$\ddot{\theta} - \lambda^2\,\theta = 0 \qquad \lambda = \sqrt{\frac{\left(\frac{1}{2}m_1 + m_2\right)g - \frac{1}{2}\,k\,l}{\left(\frac{1}{3}m_1 + m_2\right)l}}$$

$$\theta(t) = a_1\,e^{\lambda t} + a_2\,e^{-\lambda t}$$

비진동 증폭 ⇒ 불안정(unstable)